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Supplemental Material

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Supplementary Material

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Physical Understanding of Human-Induced Changes in U.S. Hot Droughts Using Equilibrium Climate Simulations

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Conditional distribution functions and the inverse forms of five candidate copulas are given as following:

1. Gaussian (Normal) Copula

The bivariate Gaussian (Normal) copula has distribution function:

$$C(u, v; \theta) = \Phi_{\theta}(\Phi^{-1}(u), \Phi^{-1}(v))$$

where u and v are the marginal distribution functions of the random variables X and Y in the range $[0, 1]$, Φ is the standard normal distribution $N(0,1)$ with mean zero and unit variance, Φ^{-1} is its inverse, and Φ_{θ} is the bivariate standard normal distribution with correlation θ .

The corresponding density function is:

$$c(u, v; \theta) = \frac{1}{\sqrt{1-\theta^2}} \exp\left[-\frac{\theta^2(u^2+v^2)-2\theta uv}{2(1-\theta^2)}\right].$$

The h and h^{-1} functions for Gaussian (Normal) copulas are:

$$h(u, v; \theta) = \Phi\left[\frac{\Phi^{-1}(u)-\theta\Phi^{-1}(v)}{\sqrt{1-\theta^2}}\right], \text{ and}$$

$$h^{-1}(u, v; \theta) = \Phi[\Phi^{-1}(u)\sqrt{1-\theta^2} + \theta\Phi^{-1}(v)]$$

The parameter space for the dependence parameter of Normal copulas is $\theta \in (-1,1)$.

2. Student- t Copula

The bivariate Student- t copula has distribution function:

$$C(u, v; \rho, \delta) = t_{\rho, \delta}[t_{\delta}^{-1}(u), t_{\delta}^{-1}(v)]$$

where $t_{\rho, \delta}$ is the bivariate Student- t distribution function with correlation parameter ρ and δ degrees of freedom, and t^{-1} denotes the inverse univariate Student- t distribution function with δ degrees of freedom.

The corresponding density function is:

$$c(u, v; \rho, \delta) = \frac{\Gamma(\frac{\delta+2}{2})\Gamma(\frac{\delta}{2})}{\sqrt{1-\rho^2} [\Gamma(\frac{\delta+1}{2})]^2} \times \frac{\{[1 + \frac{(t_{\delta}^{-1}(u))^2}{\delta}][1 + \frac{(t_{\delta}^{-1}(v))^2}{\delta}]\}^{\frac{\delta+1}{2}}}{\{1 + \frac{[t_{\delta}^{-1}(u)]^2 + [t_{\delta}^{-1}(v)]^2 - 2\rho t_{\delta}^{-1}(v)}{\delta(1-\rho^2)}\}^{\frac{\delta+2}{2}}}$$

The h and h^{-1} functions for Student- t copulas are:

$$h(u, v; \rho, \delta) = t_{\delta+1} \left\{ \frac{t_{\delta}^{-1}(u) - \rho t_{\delta}^{-1}(v)}{\sqrt{\frac{[\delta + (t_{\delta}^{-1}(v))^2] (1 - \rho^2)}{\delta + 1}}} \right\}, \text{ and}$$

$$h^{-1}(u, v; \rho, \delta) = t_{\delta} \left\{ t_{\delta+1}^{-1}(u) \sqrt{\frac{(\delta + (t_{\delta}^{-1}(v))^2 (1 - \rho^2))}{\delta + 1}} + \rho t_{\delta}^{-1}(v) \right\}$$

The parameter space for the correlation parameter is $\rho \in (-1, 1)$, and for degrees of freedom parameter is $\delta > 2$.

3. Frank Copula

The bivariate Frank copula has distribution function:

$$C(u, v; \theta) = -\theta^{-1} \log([1 - e^{-\theta} - (1 - e^{\theta u})(1 - e^{\theta v})]/(1 - e^{-\theta}))$$

The corresponding density function is:

$$c(u, v; \theta) = \frac{\theta(1 - e^{-\theta})e^{-\theta(u+v)}}{[(1 - e^{-\theta}) - (1 - e^{\theta u})(1 - e^{\theta v})]^2}.$$

The h and h^{-1} functions for Frank copulas are:

$$h(u, v; \theta) = \frac{e^{-\theta v}}{\frac{1 - e^{-\theta}}{1 - e^{-\theta u}} + e^{-\theta v} - 1}, \text{ and}$$

$$h^{-1}(u, v; \theta) = -\log \left\{ 1 - \frac{1 - e^{-\theta}}{(u^{-1} - 1)e^{-\theta v} + 1} \right\} / \theta$$

The parameter space for θ is $0 \leq \theta < \infty$.

4. Clayton Copula

The bivariate Clayton copula has distribution function:

$$C(u, v; \theta) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}$$

The corresponding density function is:

$$c(u, v; \theta) = (1 + \theta)(uv)^{-\theta-1}(u^{-\theta} + v^{-\theta} - 1)^{-2-1/\theta}.$$

The h and h^{-1} functions for Clayton copulas are:

$h(u, v; \theta) = v^{-\theta-1}(u^{-\theta} + v^{-\theta} - 1)^{-1-1/\theta}$, and

$h^{-1}(u, v; \theta) = [(uv^{\theta+1})^{-\theta(1+\theta)} + 1 - v^{-\theta}]^{-1/\theta}$

The parameter space for θ is $0 \leq \theta < \infty$.

5. Gumbel Copula

The bivariate Gumbel copula has distribution function:

$C(u, v; \theta) = \exp\{-[(-\log u)^\theta + (-\log v)^\theta]^{\frac{1}{\theta}}\}$

The corresponding density function is:

$c(u, v; \theta) = C(u, v; \theta)(uv)^{-1} \times$

$$\frac{[(\log u)(\log v)]^{\theta-1}}{[(-\log u)^\theta + (-\log v)^\theta]^{2-\frac{1}{\theta}}} \{ [(-\log u)^\theta + (-\log v)^\theta]^{\frac{1}{\theta}} + \theta - 1 \}$$

where the dependence is controlled by $\theta \geq 1$. Perfect dependence is obtained when $\theta \rightarrow \infty$, and $\theta = 1$ implies independence.

The h function for Gumbel copulas is:

$h(u, v; \theta) = v^{-1} \exp\left\{-[(-\log u)^\theta + (-\log v)^\theta]^{\frac{1}{\theta}}\right\} \left[1 + \left(\frac{\log u}{\log v}\right)^\theta\right]^{-1+1/\theta}$

There is no closed form of h^{-1} function for Gumbel copulas. Therefore, a numerical routine, i.e. Newton-Raphson method, is used to invert it.